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List-colouring the square of a K_4-minor-free graph

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Abstract

Let G be a K_4-minor-free graph with maximum degree Δ. It is known that if $\Delta \in \{2, 3\}$ then G^2 is $(\Delta+2)$-degenerate, so that $\chi(G^2) \leqslant \mathrm{ch}(G^2) \leqslant \Delta+3$. It is also known that if $\Delta \geqslant 4$ then G^2 is $(\lfloor \frac{3}{2}\Delta \rfloor + 1)$-degenerate and $\chi(G^2) \leqslant \lfloor \frac{3}{2}\Delta \rfloor + 1$. It is proved here that if $\Delta \geqslant 4$ then G^2 is $\lceil \frac{3}{2}\Delta \rceil$-degenerate and $\mathrm{ch}(G^2) \leqslant \lfloor \frac{3}{2}\Delta \rfloor + 1$. These results are sharp.

Keywords: Choosability; Minor-free graph; List square colouring

1. Introduction

We use standard terminology, as defined in the references: for example [2,5]. The *square* G^2 of a graph G has the same vertex-set as G, and two vertices are adjacent in G^2 if they are within distance two of each other in G.

There is great interest in discovering classes of graphs G for which the choosability or list chromatic number $\mathrm{ch}(G)$ is equal to the chromatic number $\chi(G)$. The *list-square-colouring conjecture* (*LSCC*) [2] is that, for every graph G, $\mathrm{ch}(G^2) = \chi(G^2)$. It is clear that this conjecture holds when the maximum degree $\Delta(G)$ of G is 0 or 1. For $\Delta(G) = 2$, it can be deduced from the results of [4]. Specifically, we can state the following, in which we say that a graph G is *cycle-k-divisible* if every cycle in G has length divisible by k.

Theorem 1. *If G is a graph with maximum degree* 2, *then*

$$\mathrm{ch}(G^2) = \chi(G^2) = \begin{cases} 3 & \text{if } G \text{ is cycle-3-divisible}, \\ 5 & \text{if } G \text{ has } C_5 \text{ as a component}, \\ 4 & \text{otherwise}. \end{cases}$$

For a K_4-minor-free graph with maximum degree $\Delta \geqslant 3$ we cannot prove that $\mathrm{ch}(G^2) = \chi(G^2)$, but we can prove the same sharp upper bound for $\mathrm{ch}(G^2)$ as for $\chi(G^2)$. Specifically, the purpose of this paper is to prove the following result, in which degeneracy(G) is the smallest integer k such that G is *k-degenerate*, that is, every subgraph of G contains a vertex with degree at most k.

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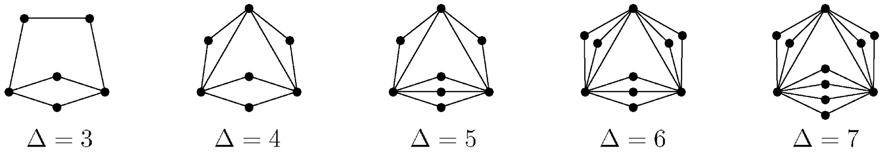

Fig. 1.

Theorem 2. *Let G be a K_4-minor-free graph with maximum degree Δ. Then*

$$\mathrm{ch}(G^2) \leqslant \begin{cases} \Delta + 3 & \text{if } \Delta = 2 \text{ or } 3, \\ \lfloor \frac{3}{2}\Delta \rfloor + 1 & \text{if } \Delta \geqslant 4, \end{cases} \tag{1}$$

and

$$\mathrm{degeneracy}(G^2) \leqslant \begin{cases} \Delta + 2 & \text{if } \Delta = 2 \text{ or } 3, \\ \lceil \frac{3}{2}\Delta \rceil & \text{if } \Delta \geqslant 4. \end{cases} \tag{2}$$

Lih et al. [3] obtained the same upper bounds as in (1) but for $\chi(G^2)$ rather than $\mathrm{ch}(G^2)$, and they gave examples to show that these bounds are sharp (see Fig. 1 for the cases $3 \leqslant \Delta \leqslant 7$). Their examples all have the property that G^2 is a complete graph. We strongly suspect that these bounds are only attained when G has a block B, of the order given in the bound, such that B^2 is complete. Theorem 1 shows that this is true when $\Delta = 2$ (since $C_5^2 \cong K_5$), but we cannot prove it in general.

In [3] the authors proved also the weaker form of (2) with $\lfloor \frac{3}{2}\Delta \rfloor + 1$ in place of $\lceil \frac{3}{2}\Delta \rceil$, and they gave examples claiming to show that it is sharp; but their examples for even values of Δ are wrong. However, their examples for odd Δ are correct, and can easily be modified to show that the bound in (2) is sharp even when Δ is even. To be specific, let $k \geqslant 2$, and let G_{2k} be formed from two nonadjacent edges uv and wx by adding $k-1$ paths of length 2 between u and v, and between w and x, and adding k paths of length 2 between u and w, and between v and x. Then G_{2k} is K_4-minor-free, and has maximum degree $2k$, and the minimum degree of G_{2k}^2 is $3k$. (For the examples when Δ is odd, given in [3], form G_{2k+1} from G_{2k} by adding a further path of length 2 between u and w, and another between v and x. Then G_{2k+1} has maximum degree $\Delta = 2k+1$, and the minimum degree of G_{2k+1}^2 is $3k+2 = \lceil \frac{3}{2}\Delta \rceil$.)

In proving Theorem 2 we will make use of the following result of Dirac [1].

Theorem 3 (*Dirac [1]*). *Every K_4-minor-free graph has a vertex with degree at most* 2.

If G is a graph such that $\Delta(G) \geqslant 3$, then G_1 will denote the graph whose vertices are the vertices that have degree at least 3 in G, where two vertices are adjacent in G_1 if and only if they are connected in G by an edge or by a path whose internal vertices all have degree 2 in G. So G_1 exists if and only if $\Delta(G) \geqslant 3$. Clearly G_1 is a minor of G. The following result is not difficult to see.

Theorem 4. *If G is a graph that does not contain a vertex with degree* 0 *or* 1 *or two adjacent vertices with degree* 2, *then G_1 exists and has no vertex with degree* 0. *If, in addition, G does not contain a* 4*-cycle $xuyvx$ such that u and v both have degree* 2 *in G, then G_1 has no vertex with degree* 1.

We will denote $(G_1)_1$ by G_2. As usual, $N(v) = N_G(v)$ will denote the set, and $d(v) = d_G(v)$ will denote the number, of vertices adjacent to v in the graph G.

2. Proof of Theorem 2

The rest of this paper is devoted to a proof of Theorem 2. Lih et al. [3] proved that if G is a K_4-minor-free graph such that $\Delta(G) = 2$ or 3, then G^2 is $(\Delta(G)+2)$-degenerate, and it follows immediately from this that $\mathrm{ch}(G^2) \leqslant \Delta(G) + 3$. Thus to prove Theorem 2 it suffices to prove the result for $\Delta(G) \geqslant 4$, which we restate as follows.

Theorem 5. *Let G be a K_4-minor-free graph with maximum degree $\Delta \geqslant 4$. Then* $\mathrm{ch}(G^2) \leqslant \lfloor \frac{3}{2}\Delta \rfloor + 1$ *and G^2 is $\lceil \frac{3}{2}\Delta \rceil$-degenerate.*

Proof. Fix the value of $\Delta \geqslant 4$, and note that $\Delta + 2 \leqslant \lfloor \frac{3}{2}\Delta \rfloor$ and $\Delta + 3 \leqslant \lfloor \frac{3}{2}\Delta \rfloor + 1$. Suppose if possible that G_{c} and G_{d} are K_4-minor-free graphs with maximum degree *at most* Δ and as few vertices as possible such that $\mathrm{ch}(G_{\mathrm{c}}^2) > \lfloor \frac{3}{2}\Delta \rfloor + 1$ and G_{d}^2 is not $\lceil \frac{3}{2}\Delta \rceil$-degenerate. Then

$$\Delta + 3 \leqslant \lfloor \tfrac{3}{2}\Delta \rfloor + 1 \leqslant \lceil \tfrac{3}{2}\Delta \rceil + 1 \leqslant \delta(G_{\mathrm{d}}^2). \tag{3}$$

Assume that every vertex v of G_{c} is given a list $L(v)$ of $\lfloor \frac{3}{2}\Delta \rfloor + 1$ colours in such a way that G_{c}^2 has no proper colouring from these lists. Let G denote G_{c} or G_{d}. We will prove various statements about G. Clearly G is connected.

Claim 1. *G does not contain a vertex of degree* 1, *or two adjacent vertices of degree* 2.

Proof. Suppose G contains a vertex u of degree 1, or two adjacent vertices v, w of degree 2. Then

$$(G - u)^2 = G^2 - u, \quad (G - \{v, w\})^2 = G^2 - \{v, w\}, \tag{4}$$

$$d_{G^2}(u) \leqslant \Delta \quad \text{and} \quad d_{G^2}(v), d_{G^2}(w) \leqslant \Delta + 2 < \lfloor \tfrac{3}{2}\Delta \rfloor + 1 \leqslant \delta(G_{\mathrm{d}}^2) \tag{5}$$

by (3). This is a contradiction if $G = G_{\mathrm{d}}$, and so we may suppose that $G = G_{\mathrm{c}}$. By the minimality of G_{c} there is a colouring of $(G - u)^2$ or $(G - \{v, w\})^2$ from its lists, and this colouring can be extended to G^2 by (4) and (5). This contradiction shows that G contains no such vertex u or vertices v, w. □

Claim 2. *The graph G_1 (defined before Theorem* 4) *exists, and has no vertex with degree* 0 *or* 1, *and at least one vertex with degree* 2.

Proof. By Theorem 4 and Claim 1, G_1 exists and has no vertex with degree 0. Suppose G_1 has a vertex u with exactly one neighbour x in G_1. Then x may or may not be a G-neighbour of u, but every G-neighbour of u different from x is a vertex of degree 2 that is adjacent to x. Thus $(G - u)^2 = G^2 - u$ and $d_{G^2}(u) \leqslant d_G(x) + 1 \leqslant \Delta + 1 < \delta(G_{\mathrm{d}}^2)$, and if $G = G_{\mathrm{c}}$ then a colouring of $G^2 - u$ from its lists can be extended to G^2. This contradiction shows that G_1 has no vertex with degree 1. Since G_1 is a minor of G and so is K_4-minor-free, it follows from Theorem 3 that G_1 must have a vertex with degree 2. This completes the proof of Claim 2. □

Before considering a vertex with degree 2 in G_1, we will consider an arbitrary vertex w with degree 2 in G. If the neighbours of w are u, v, say, let M_{uv} be the set, and m_{uv} the number, of vertices of degree 2 in G with the same neighbours u, v as w (so that $w \in M_{uv}$), and suppose there are m'_{uv} vertices of degree greater than 2 in G that are adjacent to both u and v. Let $H := G - w$ if $uv \in E(G)$ and $H := (G - w) + uv$ otherwise, so that $G^2 - w \subseteq H^2$. By (3), and since a colouring of H^2 can be extended to G^2 if $d_{G^2}(w) \leqslant \lfloor \frac{3}{2}\Delta \rfloor$, we may assume that

$$d_{G^2}(w) \geqslant \lfloor \tfrac{3}{2}\Delta \rfloor + 1 \geqslant \Delta + 3. \tag{6}$$

However,

$$d_{G^2}(w) \leqslant d_G(u) + d_G(v) - m_{uv} - m'_{uv} + 1 - 2\varepsilon_{uv}, \tag{7}$$

where $\varepsilon_{uv} = 1$ if u, v are adjacent in G and 0 otherwise. We will use this terminology in what follows.

Claim 3. *Δ is odd, say $\Delta = 2k + 1$, where $k \geqslant 2$. Also, every vertex of degree* 2 *in G_1 looks in G like vertex u of Fig.* 2, *where x and y are nonadjacent and are the only vertices in Fig.* 2 *with neighbours that are not shown, and $d_G(x) = d_G(u) = d_G(y) = \Delta = 2k + 1$ and $d_{G^2}(z) = \lceil \frac{3}{2}\Delta \rceil = 3k + 2$ for every vertex $z \in M_{ux} \cup M_{uy}$.*

Proof. It follows from Claim 2 that there is a vertex with degree 2 in G_1. Let u be any such vertex, with neighbours x, y in G_1, so that

$$d_G(u) = m_{ux} + m_{uy} + \varepsilon_{ux} + \varepsilon_{uy}. \tag{8}$$

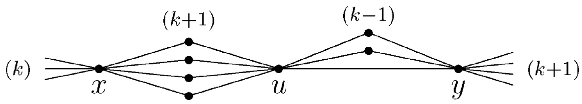

Fig. 2.

By the definition of G_1, $d_G(u)\geqslant 3$, and so m_{ux} and m_{uy} are not both zero. If $m_{ux}\neq 0$ and $w\in M_{ux}$, then (7) and (8) give

$$\begin{aligned} d_{G^2}(w) &\leqslant m_{ux}+m_{uy}+\varepsilon_{ux}+\varepsilon_{uy}+d_G(x)-m_{ux}-m'_{ux}+1-2\varepsilon_{ux} \\ &\leqslant \Delta+1+m_{uy}-\varepsilon_{ux}+\varepsilon_{uy}. \end{aligned} \tag{9}$$

If $m_{uy}=0$ and $w\in M_{ux}$, then (9) gives $d_{G^2}(w)\leqslant\Delta+2$, which contradicts (6); and the same holds by symmetry if $m_{ux}=0$ and $w\in M_{uy}$. Thus m_{ux} and m_{uy} are both nonzero. Let $w\in M_{ux}$ and $w'\in M_{uy}$. Then, by analogy with (9),

$$d_{G^2}(w')\leqslant\Delta+1+m_{ux}+\varepsilon_{ux}-\varepsilon_{uy}. \tag{10}$$

Therefore

$$\begin{aligned} \min\{d_{G^2}(w),d_{G^2}(w')\} &\leqslant \Delta+1+\tfrac{1}{2}(m_{ux}+m_{uy}) \\ &= \Delta+1+\tfrac{1}{2}(d_G(u)-\varepsilon_{ux}-\varepsilon_{uy}) \end{aligned} \tag{11}$$

by (8). It follows that $\min\{d_{G^2}(w),d_{G^2}(w')\}\leqslant\frac{3}{2}\Delta+1$.

Suppose first that $\varepsilon_{ux}=\varepsilon_{uy}=0$. Then

$$d_{G^2}(u)\leqslant d_G(u)+2\leqslant\Delta+2<\delta(G_{\mathrm{d}}^2)$$

by (3). This is a contradiction if $G=G_{\mathrm{d}}$; so suppose $G=G_{\mathrm{c}}$, and suppose w.l.o.g. $d_{G^2}(w)\leqslant d_{G^2}(w')$. Then we can colour G^2 from its lists by first colouring G^2-w, which is possible by the minimality of G_{c}, then uncolouring u, then colouring w, and finally colouring u. This contradiction shows that $\varepsilon_{ux}+\varepsilon_{uy}\geqslant 1$.

If Δ is even, then it follows from (11) that $\min\{d_{G^2}(w),d_{G^2}(w')\}\leqslant\frac{3}{2}\Delta$, which contradicts (6). So Δ must be odd, say $\Delta=2k+1$ and $\lfloor\frac{3}{2}\Delta\rfloor+1=3k+2$, where $k\geqslant 2$ since $\Delta\geqslant 4$ by the hypothesis of the theorem. In order to avoid the contradiction $\min\{d_{G^2}(w),d_{G^2}(w')\}\leqslant\lfloor\frac{3}{2}\Delta\rfloor=3k+1$, necessarily $d_G(u)=\Delta$, $\varepsilon_{ux}+\varepsilon_{uy}=1$, and equality holds in (11). Therefore equality holds in (9) and (10), and $d_{G^2}(w)=d_{G^2}(w')=3k+2=\lceil\frac{3}{2}\Delta\rceil$. Assuming w.l.o.g. that $\varepsilon_{ux}=0$ and $\varepsilon_{uy}=1$, (9) and (10) give $m_{uy}=k-1$ and $m_{ux}=k+1$. Moreover, for equality to hold in (9) and (10), necessarily $d_G(x)=d_G(y)=\Delta$ and $m'_{ux}=m'_{uy}=0$. In particular, since $m'_{ux}=0$, there is no edge xy in G. This completes the proof of Claim 3. □

Since Claim 3 contradicts (3) if $G=G_{\mathrm{d}}$, this completes the proof that G^2 is $\lceil\frac{3}{2}\Delta\rceil$-degenerate. So from now on we will assume that $G=G_{\mathrm{c}}$, and that every vertex of G has a list of $\lfloor\frac{3}{2}\Delta\rfloor+1=3k+2$ colours.

Claim 4. *G_1 does not contain two adjacent vertices with degree* 2.

Proof. Suppose it does. Then, in G, these vertices occur as u and v in Fig. 3(a) or (b), where x and y are the only vertices with neighbours that are not shown, and $x\neq y$, since the maximum degree $\Delta(G)=2k+1$ would be exceeded if $x=y$ in Fig. 3(a), and by Claim 3, x and v must not be adjacent in Fig. 3(b). Possibly x and y are adjacent, in which case x counts as one of the k or $k+1$ 'unshown' neighbours of y, and vice versa; this does not affect the following argument. Note that $M_{ux}\neq\emptyset$ since $k\geqslant 2$. If $w\in M_{ux}$ then $G^2-w=(G-w)^2$. Let us colour G^2-w from its lists, and then uncolour all the vertices in $M_{ux}\cup M_{uv}\cup M_{vy}$. For each uncoloured vertex z, let $L'(z)$ denote the 'residual list' of colours in $L(z)$ that are not used on any G^2-neighbour of z and so are still available for use on z. At this point every vertex $z\in M_{ux}\cup M_{vy}$ has $k+3$ coloured neighbours in G^2 and so $|L'(z)|\geqslant 2k-1$. Note that all the uncoloured

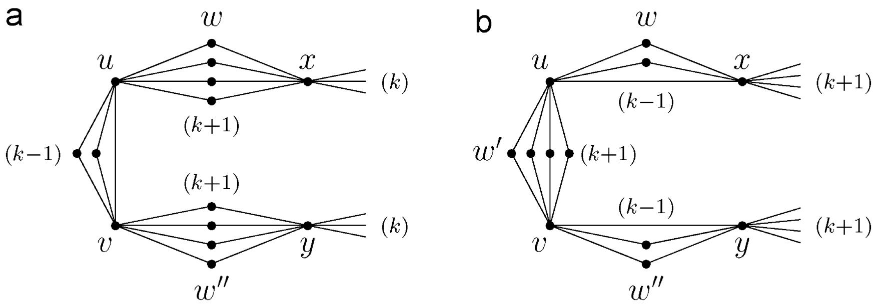

Fig. 3.

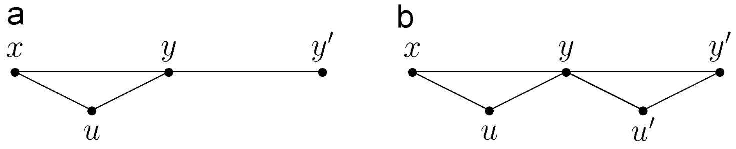

Fig. 4.

vertices have degree $\lfloor\frac{3}{2}\Delta\rfloor + 1 = 3k + 2$ in G^2, and so if we try to recolour first the vertices in M_{ux} and then those in $M_{uv} \cup M_{vy}$, it is only at the last vertex to be coloured that we may fail.

Let $w \in M_{ux}$, $w' \in M_{uv}$ and $w'' \in M_{vy}$. Then w'' has $k + 3$ coloured G^2-neighbours and $2k - 1$ uncoloured G^2-neighbours, and $|L'(w'')| \geqslant 2k - 1$. In Fig. 3(a), u has only two coloured G^2-neighbours, and so there are at least $3k$ colours in $L(u)$ that are not used on any G^2-neighbour of u. If $|L'(w'')| = 2k - 1$, which implies that the colour of u is in $L(w'')$ and is not used on any other G^2-neighbour of w'', then we can change the colour of u to make $|L'(w'')| = 2k$; then we can recolour all the vertices in M_{ux}, then M_{uv}, then M_{vy}, ending with w''. This contradiction shows that u, v must be as in Fig. 3(b). Then w' has four coloured G^2-neighbours, and so $|L'(w')| \geqslant 3k - 2$. If $L'(w) \cap L'(w'') \neq \emptyset$, then we can give w and w'' the same colour, then recolour all remaining vertices in $M_{ux} \cup M_{vy}$, and then recolour those in M_{uv}, which is possible since every vertex in M_{uv} has two G^2-neighbours with the same colour. So we may suppose that $L'(w) \cap L'(w'') = \emptyset$, so that $|L'(w) \cup L'(w'')| \geqslant 4k - 2$. Thus either $|L'(w')| \geqslant 4k - 2 > 3k - 2$, or else w or w'' can be given a colour not in $L'(w')$. In either case, the remaining vertices can now be coloured, with w' being coloured last. This contradiction completes the proof of Claim 4. □

Claim 5. *G_1 does not contain a 4-cycle $xuyvx$ in which u and v both have degree 2.*

Proof. Suppose it does. Then, by Claim 3, u would contribute k to the degree of one of x, y in G and $k + 1$ to the degree of the other in G, and so would v, so that (since $\Delta(G) = 2k + 1$) x and y could have no other neighbours. Thus x, u, y and v would all have degree 2 in G_1, and this would contradict Claim 4. □

Claim 6. *The graph $G_2 = (G_1)_1$ exists, and has no vertex with degree 0 or 1, and at least one vertex with degree 2.*

Proof. It follows from Theorem 4 and Claims 2, 4 and 5 that the graph G_2 exists and has no vertex with degree 0 or 1. Since G_2 is a minor of G_1 and hence of G, it follows from Theorem 3 that G_2 must have a vertex with degree 2. □

We will now establish a contradiction by proving the following.

Claim 7. *G_2 contains no vertex with degree 2.*

Proof. Let y be a vertex of degree 2 in G_2, with neighbours x, y'. Since y has degree at least 3 in G_1, it follows from Claims 4 and 5 that y appears in G_1 as in Fig. 4(a) or (b), where x and y' are the only vertices with neighbours that are

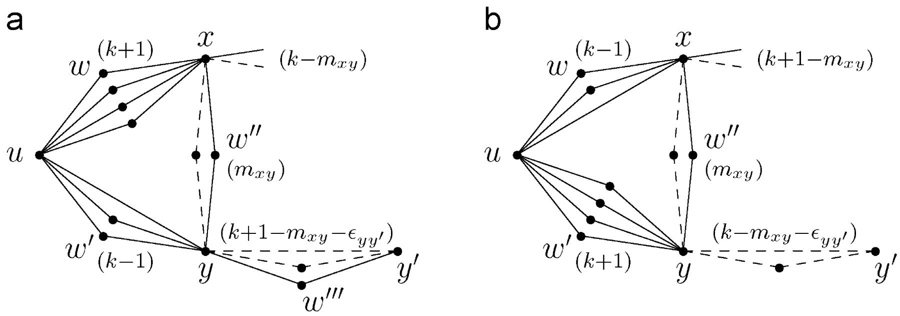

Fig. 5.

not shown. (Note that if y appears as in Fig. 4(b) but with the edge yy' missing, then it also appears as in Fig. 4(a).) However, by Claim 3, each of u and u' will contribute at least k edges towards the degree of y in G, and so if y is as in Fig. 4(b) in G_1 then $d_G(y) \geqslant 2k+2 > \Delta$, a contradiction. So y occurs in G_1 as in Fig. 4(a), and so by Claim 3 it occurs in G as in Fig. 5(a) or (b), where x and y' are the only vertices with neighbours that are not shown, and $\varepsilon_{yy'} = 1$ if there is an edge yy' in G and 0 otherwise. (Possibly one of the edges from x to an 'unshown' neighbour actually goes to y'; this does not affect the following argument.) There is at least one edge of G from x to an unshown vertex (or to y'), since otherwise u and x are adjacent vertices of degree 2 in G_1, contrary to Claim 4. Thus $m_{xy} \leqslant k-1$ in Fig. 5(a), and the same is true in Fig. 5(b) since otherwise $d_G(y)$ would be at least $2k+2 > \Delta$ in view of the edge between y and y' in G_1. This implies that $M_{yy'} \neq \emptyset$ in Fig. 5(a) (but possibly $M_{yy'} = \emptyset$ in Fig. 5(b)). Since we know that x and y are adjacent in G_1 by Fig. 4(a) but not in G by Claim 3, there must be at least one vertex of degree 2 between them in G. Thus $1 \leqslant m_{xy} \leqslant k-1$.

Let $w \in M_{ux}$, $w' \in M_{uy}$ and $w'' \in M_{xy}$, and in case (a) let $w''' \in M_{yy'}$. Colour $G^2 - w'$ from its lists, and then uncolour all the vertices in

$$\{u, y\} \cup M_{ux} \cup M_{uy} \cup M_{xy} \cup M_{yy'},$$

leaving x and y' coloured. For each uncoloured vertex z, let $L'(z)$ denote the residual list of colours that can be used on z. Since $|L(z)| = 3k+2 = d_{G^2}(z)$ for every vertex $z \in M_{ux} \cup M_{uy}$, there is no loss of generality in assuming that $|L'(z)|$ is equal to the number of uncoloured G^2-neighbours of z, for all such z. In particular, since x is a G^2-neighbour of w' in case (b) but not in case (a), we may assume that

$$|L'(w')| = \begin{cases} 3k+2-\varepsilon_{yy'} & \text{in case (a),} \\ 3k+1-\varepsilon_{yy'} & \text{in case (b).} \end{cases}$$

We are going to recolour all the uncoloured vertices and then colour w' last. To do this, we will colour two G^2-neighbours of w' (w and w''' in case (a), and w and y in case (b)) so that either they have the same colour, or one of them has a colour not in $L'(w')$. Note that, since w can be given any colour not used on x or a G-neighbour of x,

$$|L'(w)| \geqslant \begin{cases} (3k+2)-(k+1-m_{xy}) = 2k+1+m_{xy} & \text{in case (a),} \\ (3k+2)-(k+2-m_{xy}) = 2k+m_{xy} & \text{in case (b).} \end{cases}$$

In case (a), $w''' \in N_{G^2}(w') \backslash N_{G^2}(w)$. Now, $y' \in N_{G^2}(w''')$, and y' has $k+1-m_{xy}$ uncoloured G-neighbours (including y if $\varepsilon_{yy'} = 1$), hence at most $\Delta - (k+1-m_{xy})$ G-neighbours that are already coloured. Therefore

$$|L'(w''')| \geqslant (3k+2) - 1 - \Delta + (k+1-m_{xy}) = 2k+1-m_{xy},$$

so that $|L'(w)| + |L'(w''')| \geqslant 4k+2 > |L'(w')|$.

In case (b), $y \in N_{G^2}(w') \backslash N_{G^2}(w)$. Now, $x, y' \in N_{G^2}(y)$, and y' has at most $\Delta - (k-m_{xy})$ G-neighbours that are already coloured, and so the number of coloured G-neighbours of y' that are in $N_{G^2}(y) \backslash \{x\}$ is at most

$$\varepsilon_{yy'}[\Delta - (k-m_{xy})] \leqslant (\varepsilon_{yy'} - 1) + \Delta - (k-m_{xy}).$$

Thus

$$|L'(y)| \geqslant (3k+2) - 2 - (\varepsilon_{yy'} - 1) - \Delta + (k - m_{xy}) = 2k - \varepsilon_{yy'} - m_{xy},$$

so that $|L'(w)| + |L'(y)| \geqslant 4k - \varepsilon_{yy'} > |L'(w')|$.

Let $z := w'''$ in case (a) and $z := y$ in case (b), so that $|L'(w)| + |L'(z)| > |L'(w')|$ in each case. Colour w and z so that either they have the same colour or one of them has a colour that is not in $L'(w')$. In case (a), we can now recolour all remaining vertices of $M_{yy'}$ and then y, which is adjacent in G^2 to at most $\Delta + 2$ coloured vertices, namely w, x, y', and at most $\Delta - 1$ coloured G-neighbours of y'. In case (b), we can recolour all vertices (if any) in $M_{yy'}$, the last of which to be recoloured is adjacent in G^2 to at most $\Delta + 1$ coloured vertices, namely y, y', and at most $\Delta - 1$ coloured G-neighbours of y'.

We can now recolour all vertices in $M_{xy} \cup \{u\}$, since the last of these to be coloured will be adjacent in G^2 to at most $3k + 1$ coloured vertices, namely w, x, y, and vertices that are adjacent to x and y in G, a total of at most $3 + (2k+1) - m_{xy} \leqslant 2k + 3 \leqslant 3k + 1$. Finally, we can recolour all the vertices in $M_{ux} \cup M_{uy}$ that are still uncoloured, ending with w', since each of these vertices has degree $3k + 2$ in G^2, and w' either has two G^2-neighbours with the same colour or has a G^2-neighbour w or z with a colour not in $L'(w')$. Thus all vertices of G^2 can be coloured from their lists, and this contradiction completes the proof of Claim 7. □

Claim 7 contradicts Claim 6, and this contradiction completes the proof of Theorem 5. □

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